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Irregular frequencies and the motion of floating bodies

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A floating horizontal cylinder of infinite length is performing simple harmonic oscillations of small amplitude in the free surface of a uniform inviscid fluid under gravity. The cylinder intersects the mean free surface at right angles, and the fluid is bounded below by a fixed horizontal plane. Let the corresponding two-dimensional velocity potential be expressed as a distribution of simple wave sources over the boundary of the body. Then it is known that the source density satisfies a Fredholm integral equation of the second kind which has a unique solution except at those frequencies (the irregular frequencies) at which the Fredholm determinant vanishes. The present work is concerned with the irregular frequencies. Let the simple wave sources be replaced by a fundamental solution which consists of a simple wave source together with additional wave singularities inside the cylinder. It is shown how irregular frequencies can be eliminated by an appropriate choice of these singularities.

1. Introduction

Consider a floating horizontal cylinder of infinite length which is performing simple-harmonic small oscillations in the free surface of a uniform inviscid fluid under gravity. It will be assumed throughout the present work that the cylinder intersects the free surface at right angles, and that the fluid is bounded below by a fixed horizontal plane. The mathematical theory of this two-dimensional wave motion leads to a linearized potential problem which was treated in a classical paper by F. John (1950).

Two questions need to be discussed. First there is the question of uniqueness: John showed that there is at most one solution unless at the given frequency there are exterior eigensolutions (bound states or trapping modes). He also showed that there are no bound states for bodies satisfying a certain geometrical condition. It is widely believed that bound states never occur and that John's geometrical condition is not necessary, but this has not yet been shown. In the present work it will always be assumed that there are no bound states at the frequency of oscillation.

Secondly there is the question of existence. John represented the potential by a distribution of simple wave sources over the body. The source density was found to satisfy a Fredholm integral equation of the second kind which has a unique solution except possibly at the discrete set of frequencies (the irregular frequencies) at which the Fredholm determinant vanishes. A separate and rather complicated argument was required for the irregular frequencies; the boundary-value problem was again found to have a unique solution which cannot however be represented as a distribution of simple sources. There is an infinite discrete set of irregular frequencies, corresponding to the eigenfrequencies of an interior problem.

Irregular frequencies give rise to computational as well as theoretical difficulties and have accordingly been much studied, particularly in acoustics where bound states are known not to exist. (For references see, for example, Jones 1974.) Irregular frequencies have no physical meaning but are associated with a given representation; when the simple wave source in John's representation is replaced by a different fundamental solution the irregular values are in general modified and possibly removed altogether. Some earlier work on the short-wave asymptotics of a half-immersed circle illustrates this property (Ursell 1953, 1961). Additional multipole singularities were placed at the centre in order to make the kernel of the resulting integral equation small. This construction thus incidentally removed all the high irregular frequencies. More recently the lower irregular frequencies have been studied, and it has been shown (see § 4 below) that some of these can be removed in some cases by placing additional singularities inside the body.

In the present work this method is studied more systematically. The simple wave source $G_0(P,Q)$, defined in (2.8) below, is replaced by a fundamental solution $G_1(P,Q)$ which has additional multipole wave singularities inside the body. The argument is an adaptation of the work of Jones (1974) and Ursell (1978) for the exterior acoustic problem but is more complicated because the multipole singularities are more complicated for water waves than for sound waves. It will be seen, nevertheless, that the result takes a simple form.

2. The boundary-value problem

Let co-ordinates be taken with the origin O in the mean free surface of the fluid, with the x axis horizontal and normal to the generators of the cylinder, and with the y axis vertical (y increasing with depth). In the (x,y) plane, let the fluid domain be denoted by D, the boundary of the cylinder by ∂D, the mean free surface by F, the interior of the cylinder by D_-. It is assumed that the origin lies in D_-. Let points in D be denoted by capital letters P, Q, points on ∂D by small letters p, q. Let $y = h$ be the lower boundary of the fluid.

Then the fluid motion is described by a velocity potential $\mathrm{Re}\{\phi(x,y)\,e^{-i\sigma t}\}$. By hypothesis the curve ∂D intersects $y = 0$ at right angles, the normal velocity $V(p)$ is prescribed on ∂D, also the boundary conditions on ∂D and F are linearized. The time factor $e^{-i\sigma t}$ will always be omitted. Then the function $\phi(x,y)$ satisfies the equation of continuity

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi(x,y) = 0 \quad \text{in } D, \tag{2.1}$$

with the boundary conditions

$$K\phi+\frac{\partial\phi}{\partial y} = \frac{\sigma^2}{g}\phi+\frac{\partial\phi}{\partial y} = 0 \quad \text{on } F, \tag{2.2}$$

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = h, \tag{2.3}$$

and

$$\frac{\partial\phi}{\partial n} = V(p) \quad \text{on } \partial D, \tag{2.4}$$

where $V(p)$ is a prescribed function of position, and $\partial/\partial n$ denotes differentiation along the normal from ∂D into D. At infinity there is the radiation condition that waves travel outwards to infinity.

Let $G(x,y;\,\xi,\eta) = G(P,Q)$ denote a fundamental solution with a source singularity at $(x,y) = (\xi,\eta)$, satisfying

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) G(x,y;\,\xi,\eta) = 0 \text{ in } D, \tag{2.5}$$

$$KG+\frac{\partial G}{\partial y} = 0 \text{ on } F, \tag{2.6}$$

$$\frac{\partial G}{\partial y} = 0 \quad \text{on} \quad y = h, \tag{2.7}$$

$G(x,y;\,\xi,\eta) - \frac{1}{2}\log\{(x-\xi)^2+(y-\eta)^2\}$ is bounded near (ξ,η), together with the radiation condition at infinity. The simplest fundamental solution is the simple wave source (Thorne 1953)

$$\begin{aligned} G_0(x,y;\,\xi,\eta;\,h) = {}& \tfrac{1}{2}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2} \\ & -2\int_0^\infty \frac{\cosh k(h-y)\cosh k(h-\eta)\cos k(x-\xi)\,dk}{\cosh kh(k\sinh kh - K\cosh kh)} \\ & -2\int_0^\infty e^{-kh}\frac{\sinh ky\sinh k\eta}{k\cosh kh}\cos k(x-\xi)\,dk, \end{aligned} \tag{2.8}$$

where, in order to satisfy the radiation condition, the path of integration is made to pass below the pole $k = k_0$ of the integrand, at which

$$K\cosh k_0 h = k_0 \sinh k_0 h.$$

It will be shown later, in appendix A, that near $(x,y) = (0,0)$ there is an expansion

$$G_0(x,y;\,\xi,\eta;\,h) = \sum_{m=0}^{\infty} \alpha_m(x,y)\,\Phi_m(\xi,\eta;\,h) \tag{2.9}$$

$$+\sum_{m=0}^{\infty} \beta_m(x,y)\,\Psi_m(\xi,\eta;\,h) \tag{2.10}$$

where the functions $\alpha_m(\;)$, $\beta_m(\;)$, and the multipole potentials $\Phi_m(\;)$, $\Psi_m(\;)$ are defined in appendix A. These functions satisfy the free-surface condition, and $\Phi_m(\;)$ and $\Psi_m(\;)$ also satisfy the bottom boundary condition and the radiation condition at infinity. A more general fundamental solution, symmetrical in (x,y) and (ξ,η), is

$$\begin{aligned} G_1(x,y;\,\xi,\eta;\,h) = {}& G_0(x,y;\,\xi,\eta;\,h) + \sum_0^M a_m\,\Phi_m(x,y;\,h)\,\Phi_m(\xi,\eta;\,h) \\ & + \sum_0^N b_m\,\Psi_m(x,y;\,h)\,\Psi_m(\xi,\eta;\,h), \end{aligned} \tag{2.11}$$

where a_m $(m = 0,1,2,\ldots,M)$ and b_m $(m = 0,1,2,\ldots,N)$ are complex-valued constants, and $\Phi_m(\;)$ and $\Psi_m(\;)$ are again the multipole potentials defined in appendix A. Thus $G_1(\;)$ has a singularity at the origin.

Let a solution $\phi(P)$ of the boundary-value problem (2.1)–(2.4) be sought in the form of a distribution of sources $G_1(\;)$ over ∂D:

$$\phi(P) = \int_{\partial D} \mu(q)\, G_1(P, q)\, ds_q, \tag{2.12}$$

where the source density $\mu(q)$ is to be determined. (Note that John (1950) sought a solution in the form of a distribution of sources $G_0(\;)$ over ∂D.) Then the boundary condition (2.4) is satisfied if $\mu(q)$ satisfies the Fredholm integral equation of the second kind

$$\pi\mu(p) + \int_{\partial D} \mu(q) \frac{\partial}{\partial n_p} G_1(p, q)\, ds_q = V(p), \tag{2.13}$$

where $\partial/\partial n_p$ denotes normal differentiation for the variable P from ∂D into D. As is well known (see, for example, John 1950) the equation (2.13) has a unique solution except at those values of the wavenumber K (the irregular values) at which the Fredholm determinant vanishes.

3. Irregular values

Suppose now that K is an irregular wavenumber; it is still assumed (see § 1 above) that at this wavenumber there are no bound states. It follows from Fredholm's theory that the homogeneous equation

$$\pi\mu(p) + \int_{\partial D} \mu(q) \frac{\partial}{\partial n_p} G_1(p, q)\, ds_q = 0, \tag{3.1}$$

then has a non-trivial solution, also denoted by $\mu(q)$. The following theorem is typical.

Theorem 1. *In the equation* (2.11) *defining* $G_1(P, Q)$, *suppose that the imaginary parts of the coefficients* $a_0, \ldots, a_M$ *and* $b_0, \ldots, b_N$ *are all strictly positive. Then every solution of the integral equation*

$$\pi\mu(p) + \int_{\partial D} \mu(q) \frac{\partial}{\partial n_p} G_1(p, q)\, ds_q = 0,$$

is a solution of

$$\pi\mu(p) + \int_{\partial D} \mu(q) \frac{\partial}{\partial n_p} G_0(p, q)\, ds_q = 0, \tag{3.2}$$

which also satisfies

$$A_m \equiv \int_{\partial D} \mu(q)\, \Phi_m(q)\, ds_q = 0, \quad m = 0, \ldots, M \tag{3.3}$$

and

$$B_m \equiv \int_{\partial D} \mu(q)\, \Psi_m(q)\, ds_q = 0, \quad m = 0, \ldots, N. \tag{3.4}$$

The proof proceeds along the same lines as in Jones (1974) and Ursell (1978) and will be given later. We begin by proving a second theorem:

Theorem 2. *Suppose that the integral equation* (3.1) *has a non-trivial solution* $\mu(q)$. *Then the interior potential*

$$u(P_-) = \int_{\partial D} \mu(q)\, G_1(P_-, q)\, ds_q, \tag{3.5}$$

defined for $P_- \in D_-$, *vanishes on* ∂D.

Proof. Consider the exterior potential

$$u(P) = \int_{\partial D} \mu(q)\, G_1(P, q)\, ds_q, \tag{3.6}$$

where $P \in D$. Evidently $u(P)$ satisfies (2.1), (2.2), (2.3) and the radiation condition, and on account of (3.1) it also satisfies $\partial u/\partial n = 0$ on D. Thus, by the assumed uniqueness property, $u(P) \equiv 0$ in D, and in particular $u(p) = 0$ on ∂D. Consider now the interior potential $u(P_-)$, defined by (3.5). This is not the analytic continuation of $u(P)$, since there is a source distribution along the common boundary ∂D, but it is known that a potential is continuous across a source distribution. Thus $u(p_-) = u(p) = 0$. This concludes the proof of theorem 2.

It does not however follow that $u(P_-)$ defined by (3.5) is an eigenfunction of the interior problem since $G_1(P_-, q)$ has singularities at the origin. In fact the expansion near the origin is given by a third theorem.

THEOREM 3. *Suppose that P_- lies inside a semicircular arc C which is small enough to lie inside D_- and which has its centre at O. Then*

$$u(P_-) = \sum_0^\infty A_m \alpha_m(P_-) + \sum_0^\infty B_m \beta_m(P_-) + \sum_0^M a_m A_m \Phi_m(P_-; h) + \sum_0^N b_m B_m \Psi_m(P_-; h), \tag{3.7}$$

where

$$A_m = \int_{\partial D} \mu(q)\, \Phi_m(q; h)\, ds_q, \quad B_m = \int_{\partial D} \mu(q)\, \Psi_m(q; h)\, ds_q,$$

where a_m and b_m are the coefficients in (2.11), *and where the wave potentials $\alpha_m(\ )$, $\beta_m(\ )$, $\Phi_m(\ )$, $\Psi_m(\ )$ are defined in appendix A.*

Proof of theorem 3. From (3.5) and (2.11) we have

$$u(P_-) = \int_{\partial D} \mu(q)\, G_0(P_-, q)\, ds_q + \Sigma a_m A_m \Phi_m(P_-) + \Sigma b_m B_m \Psi_m(P_-),$$

and, on substituting

$$G_0(P_-, q) = \Sigma \alpha_m(P_-)\, \Phi_m(q) + \Sigma \beta_m(P_-)\, \Psi_m(q),$$

(see appendix A), we have equation (3.7). This concludes the proof of theorem 3.

Following the argument of Ursell (1978), we now consider the integral

$$[u, u^*] = \int_C \left(u \frac{\partial u^*}{\partial n} - u^* \frac{\partial u}{\partial n} \right) ds,$$

where the asterisk denotes the complex conjugate. From Green's theorem, since u and u^* are both harmonic and satisfy the free-surface condition (2.2), it follows that

$$\int_C \left(u \frac{\partial u^*}{\partial n} - u^* \frac{\partial u}{\partial n} \right) ds = \int_{\partial D} \left(u \frac{\partial u^*}{\partial n} - u^* \frac{\partial u}{\partial n} \right) ds = 0,$$

since $u = u^* = 0$ on ∂D, by theorem 2. Now using theorem A 1 in appendix A and theorem B 3 in appendix B we find that

$$0 = \frac{i}{4\pi} [u, u^*] = \mathscr{H}(A_m, B_m; a_m, b_m), \tag{3.8}$$

where by definition

$$\mathscr{H}(\ ) = \sum_0^M \mathrm{Im}\,(a_m)\,|A_m|^2 + \sum_0^N \mathrm{Im}\,(b_m)\,|B_m|^2$$
$$+\frac{\pi}{2k_0h+\sinh 2k_0h}\left|a_0A_0\cosh k_0h + \frac{1}{\cosh k_0h}\sum_1^M \frac{a_mA_mk_0^{2m}}{(2m-1)!}\right|^2$$
$$+\frac{\pi k_0^2}{2k_0h+\sinh 2k_0h}\left|b_0B_0\cosh k_0h + \frac{1}{\cosh k_0h}\sum_1^N \frac{b_mB_mk_0^{2m}}{(2m)!}\right|^2. \tag{3.9}$$

We can now proceed to the

Proof of theorem 1. Since $\mathrm{Im}\,(a_m) > 0$ $(m = 0, 1, \ldots, M)$, and $\mathrm{Im}\,(b_m) > 0$ $(m = 0, 1, \ldots, N)$, we see from (3.9) that $\mathscr{H}(\ ) > 0$ except when $A_m = B_m = 0$; but, from (3.8), $\mathscr{H} = 0$. Thus we must have $A_m = B_m = 0$. It then follows that

$$\int_{\partial D} \mu(q)\frac{\partial}{\partial n_p}G_1(p,q)\,ds_q = \int_{\partial D} \mu(q)\frac{\partial}{\partial n_p}G_0(p,q)\,ds_q,$$

and therefore $\mu(q)$ satisfies the integral equation (3.2). This concludes the proof of theorem 1.

Note. The proof remains valid under wider conditions. It is sufficient that the coefficients a_m and b_m be chosen so that $\mathscr{H}(\ )$ is a positive definite Hermitian form in A_m and B_m. Also, if a certain term, a_l say, is absent in the sum (2.11), the conclusion $A_m = 0$ remains valid for $m \neq l$, although we can no longer conclude that $A_l = 0$.

Suppose, then, that we have concluded that $A_m = 0$ $(m = 0, \ldots, M)$, and $B_m = 0$ $(m = 0, \ldots, N)$. It follows from (3.7) that, near O,

$$u(P_-) = \sum_{M+1}^{\infty} A_m\alpha_m(P_-) + \sum_{N+1}^{\infty} B_m\beta_m(P_-) \tag{3.10}$$

and has no singularity at O. Thus, since from theorem 2 we have $u(p_-) = 0$, it follows that either $u(P_-)$ vanishes identically, or that it is an eigenfunction of the interior problem with the boundary conditions

$$\left(K+\frac{\partial}{\partial y}\right)u(P_-) = 0 \quad \text{on} \quad y = 0 \quad \text{and} \quad u(p_-) = 0 \quad \text{on } \partial D.$$

The latter possibility can be excluded by choosing M and N large enough. For the expansion (3.10) implies that at the origin O

$$\left(\frac{\partial}{\partial y}\right)^m u = 0, \quad m = 0, 1, 2, \ldots, 2M+1,$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)^m u = 0, \quad m = 0, 1, 2, \ldots, 2N+2,$$

but none of the eigenfunctions of (3.2) will satisfy all these conditions when M and N are chosen large enough. Let this be done, then we must have $u(P_-) \equiv 0$. Thus

$$\frac{\partial u}{\partial n}(p_-) = -\pi\mu(p) + \int_{\partial D} \mu(q)\frac{\partial}{\partial n_p}G_1(p,q)\,ds_q = 0,$$

and from (3.1) we have

$$\pi\mu(p) + \int_{\partial D} \mu(q)\frac{\partial}{\partial n_p}G_1(p,q)\,ds_q = 0.$$

Thus $\mu(p) = 0$ on ∂D. Thus the value of K under consideration is not an irregular wavenumber when M and N are large enough. It should not be difficult to estimate appropriate bounds for M and N when K is prescribed (cf. Jones 1974, § 3), but this problem will not be pursued here.

4. Applications and discussion

Sayer (1980) has made numerical calculations for some bodies symmetrical about $x = 0$. Using a symmetrical pair of source functions $G_0(\ )$, he found the expected difficulties near the first irregular frequency, which is the eigenfrequency of the first symmetrical mode of the interior problem. These difficulties could be removed by using a symmetrical pair of fundamental solutions

$$G_1(P, Q) = G_0(P, Q) + a_0 \Phi_0(P) \Phi_0(Q), \tag{4.1}$$

for suitable values of a_0; the actual choice of a_0 was found to be not critical. It was found that the same construction also removed the second and third irregular frequency. (It will be recalled, see § 1 above, that in earlier work on high-frequency asymptotics the choice of a_0 was critical.) More recently P. Martin (1980) has studied a symmetrical pair of fundamental solutions

$$G_1(P, Q) = G_0(P, Q) + a_1 \Phi_1(P) \Phi_1(Q), \tag{4.2}$$

for a symmetrical body. This is simpler than (4.1) because (particularly for $h = \infty$) Φ_1 has a simpler form than Φ_0. He found that the first irregular frequency was removed by an appropriate choice of a_1, but that numerical difficulties still occurred near the eigenfrequencies of higher symmetrical modes. Let us examine this phenomenon in the light of theorem 1. According to our argument (see the note following the proof of theorem 1), at an irregular value associated with (4.2) the corresponding interior eigenfunction must have the form

$$u(P_-) = -2A_0 e^{-Ky} \cos Kx + A_2 \alpha_2(x, y) + A_3 \alpha_3(x, y) + \dots,$$

where the term involving $\alpha_1(x, y)$ is absent. Thus

$$u(P_-) = -2A_0 e^{-Ky} \cos Kx + O((Kr)^4).$$

It is unlikely that any interior eigenfunction has precisely this form but it is physically obvious (see, for example, Ursell 1974) that the higher even eigenfunctions have nearly the form $e^{-Ky} \cos Kx$. Thus the Fredholm determinant, while not zero, would be expected to be small, and this would help to explain the computational difficulties.

In practice it is the potential on the body rather than the source strength that is required. This is given by the integral equation

$$\pi u(p) + \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_1(p, q)\, ds_q = \int_{\partial D} V(q)\, G_1(p, q)\, ds_q, \tag{4.3}$$

which is obtained from Green's theorem (cf. Ursell 1978, § 3). The kernel of this equation is the transpose of our previous equation since $G_1(P, Q) = G_1(Q, P)$ and thus has the same Fredholm determinant. Thus our theorems remain applicable to (4.3).

We can now see how John's treatment of the boundary-value problem can be simplified At an irregular frequency associated with $G_0(P, Q)$ use the fundamental solution $G_1(P, Q)$ with sufficiently large values of M and N. (See the end of § 3 above.)

Then this frequency is not an irregular frequency associated with $G_1(P,Q)$, and the ordinary theory remains applicable.

Appendix A. Expansion of the source potential

The time factor $e^{-i\sigma t}$ is omitted throughout. The wave source at (ξ, η) in water of depth h is given by the potential

$$G_0(x,y;\xi,\eta;h) = \tfrac{1}{2}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}$$
$$-2\oint_0^\infty \frac{\cosh k(h-y)\cosh k(h-\eta)\cos k(x-\xi)\,dk}{\cosh kh(k\sinh kh - K\cosh kh)}$$
$$-2\int_0^\infty e^{-kh}\frac{\sinh ky\sinh k\eta}{k\cosh kh}\cos k(x-\xi)\,dk, \qquad \text{(A 1)}$$

see Thorne (1953). We need the expansion of G_0 near $(x,y) = (0,0)$; this will be given in theorem A 1 below and involves the following singular potentials (see Yu & Ursell 1961; Ursell 1976):

(1) Wave source at $(0,0)$:

$$\Phi_0(x,y;h) = \oint_0^\infty \frac{\cosh k(h-y)\cos kx\,dk}{k\sinh kh - K\cosh kh}. \qquad \text{(A 2)}$$

(2) Horizontal wave dipole at $(0,0)$:

$$\Psi_0(x,y;h) = -\frac{\partial}{\partial x}\Phi_0, \quad \text{where } \Phi_0 \text{ is given by (A 2).} \qquad \text{(A 3)}$$

(3) Symmetrical potentials singular at $(0,0)$:

$$\Phi_m(x,y;h) = \frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}$$
$$-\frac{1}{(2m-1)!}\oint_0^\infty \frac{e^{-kh}(K+k)(K\sinh ky - k\cosh ky)\,k^{2m-2}\cos kx\,dk}{k\sinh kh - K\cosh kh}$$
$$(m = 1,2,3,\ldots). \qquad \text{(A 4)}$$

(4) Anti-symmetrical potentials singular at $(0,0)$:

$$\Psi_m(x,y;h) = -\frac{1}{2m}\frac{\partial}{\partial x}\Phi_m$$
$$= \frac{\sin(2m+1)\theta}{r^{2m+1}} + \frac{K}{2m}\frac{\sin 2m\theta}{r^{2m}}$$
$$-\frac{1}{(2m)!}\oint_0^\infty \frac{e^{-kh}(K+k)(K\sinh ky - k\cosh ky)\,k^{2m-1}\sin kx\,dk}{k\sinh kh - K\cosh kh}$$
$$(m = 1,2,3,\ldots). \qquad \text{(A 5)}$$

As $x \to +\infty$, we have

$$\Phi_0 \sim \frac{2\pi i \cosh k_0 h}{2k_0 h + \sinh 2k_0 h} \cosh k_0(h-y)\, e^{ik_0 x},$$

$$\Psi_0 \sim \frac{2\pi k_0 \cosh k_0 h}{2k_0 h + \sinh 2k_0 h} \cosh k_0(h-y)\, e^{ik_0 x},$$

$$\Phi_m \sim \frac{2\pi i}{(2m-1)!} \frac{k_0^{2m}}{\cosh k_0 h(2k_0 h + \sinh 2k_0 h)} \cosh k_0(h-y)\, e^{ik_0 x} \quad (m = 1, 2, 3, \ldots),$$

$$\Psi_m \sim \frac{2\pi}{(2m)!} \frac{k_0^{2m+1}}{\cosh k_0 h(2k_0 h + \sinh 2k_0 h)} \cosh k_0(h-y)\, e^{ik_0 x} \quad (m = 1, 2, 3, \ldots).$$

When $h = \infty$ we readily see that

$$\Phi_0(x, y; \infty) = \oint_0^\infty e^{-ky} \cos kx \frac{dk}{k-K} \sim \pi i\, e^{-Ky+iKx};$$

$$\Psi_0(x, y; \infty) = \oint_0^\infty k\, e^{-ky} \sin kx \frac{dk}{k-K} \sim \pi K\, e^{-Ky+iKx};$$

$$\Phi_m(x, y; \infty) = \frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1} \frac{\cos(2m-1)\theta}{r^{2m-1}} \quad (m = 1, 2, 3, \ldots)$$

$$= \frac{1}{(2m-1)!} \int_0^\infty k^{2m-2}(k+K)\, e^{-ky} \cos kx\, dk; \tag{A 6}$$

$$\Psi_m(x, y; \infty) = \frac{\sin(2m+1)\theta}{r^{2m+1}} + \frac{K}{2m} \frac{\sin 2m\theta}{r^{2m}} \quad (m = 1, 2, 3, \ldots)$$

$$= \frac{1}{(2m)!} \int_0^\infty k^{2m-1}(k+K)\, e^{-ky} \sin kx\, dk. \tag{A 7}$$

We note that (A 6) and (A 7) are wavefree potentials. We also define the regular harmonic wave potentials

$$\alpha_0(x, y) = -2\, e^{-Ky} \cos Kx, \quad \beta_0(x, y) = -2\, e^{-Ky} K^{-1} \sin Kx,$$

$$\alpha_m(x, y) = \frac{-2(2m-1)!}{K^{2m}} \sum_{s=2m}^{\infty} (-1)^s \frac{(Kr)^s}{s!} \cos s\theta \quad (m = 1, 2, 3, \ldots),$$

$$\beta_m(x, y) = \frac{2(2m)!}{K^{2m+1}} \sum_{s=2m+1}^{\infty} (-1)^s \frac{(Kr)^s}{s!} \sin s\theta \quad (m = 1, 2, 3, \ldots).$$

We can now state two theorems on the expansion of the wave source $G_0(\ )$.

THEOREM A 1. *When* $|x+iy| < |\xi + i\eta|$, *the source potential* $G_0(x, y; \xi, \eta; h)$ *defined by* (A 1) *can be expanded in the form*

$$G_0(x, y; \xi, \eta; h) = \sum_{m=0}^{\infty} \alpha_m(x, y)\, \Phi_m(\xi, \eta; h) + \sum_{m=0}^{\infty} \beta_m(x, y)\, \Psi_m(\xi, \eta; h).$$

THEOREM A 2. (*The special case $h = \infty$.) When $|x+iy| < |\xi+i\eta|$, the source potential $G_0(x, y;\, \xi, \eta;\, \infty)$ can be expanded in the form*

$$G_0(x,y;\,\xi,\eta;\,\infty) = \sum_{m=0}^{\infty} \alpha_m(x,y)\,\Phi_m(\xi,\eta;\,\infty) + \sum_{m=0}^{\infty} \beta_m(x,y)\,\Psi_m(\xi,\eta;\,\infty). \tag{A 8}$$

It will be convenient to begin by proving theorem A 2, and to assume initially that $y < \eta$. We have

$$G_0(x,y;\,\xi,\eta;\,\infty) = \tfrac{1}{2}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\xi)^2} - 2\int_0^\infty e^{-k(y+\eta)}\cos k(x-\xi)\frac{dk}{k-K}$$

$$= \int_0^\infty \frac{e^{-k(y+\eta)}-e^{-k|y-\eta|}}{k}\cos k(x-\xi)\,dk - 2\int_0^\infty e^{-k(y+\eta)}\cos k(x-\xi)\frac{dk}{k-K},$$

whence

$$G_0(x,y;\,\xi,\eta;\,\infty) - \alpha_0(x,y)\,\Phi_0(\xi,\eta;\,\infty) - \beta_0(x,y)\,\Psi_0(x,y;\,\infty)$$

$$= \int_0^\infty \frac{e^{-k(y+\eta)}-e^{-k(\eta-y)}}{k}\cos k(x-\xi)\,dk - 2\int_0^\infty e^{-k(y+\eta)}\cos k(x-\xi)\frac{dk}{k-K}$$

$$+2e^{-Ky}\cos Kx\int_0^\infty e^{-k\eta}\cos k\xi\frac{dk}{k-K} + 2e^{-Ky}\frac{\sin Kx}{K}\int_0^\infty k\,e^{-k\eta}\sin k\xi\frac{dk}{k-K}$$

$$= -2\int_0^\infty e^{-k\eta}\cos k\xi\left(\frac{\sinh ky}{k}\cos kx + \frac{e^{-ky}\cos kx - e^{-Ky}\cos Kx}{k-K}\right)dk \tag{A 9}$$

$$-2\int_0^\infty e^{-k\eta}\,k\sin k\xi\left(\frac{\sinh ky}{k^2}\sin kx + \frac{e^{-ky}\,k^{-1}\sin kx - e^{-Ky}\,K^{-1}\sin Kx}{k-K}\right)dk. \tag{A 10}$$

Let us consider (A 9). It is known that

$$e^{\pm ky}\cos kx = \sum_0^\infty \frac{(\pm kr)^s}{s!}\cos s\theta;$$

thus

$$\left(\frac{\sinh ky}{k}\cos kx + \frac{e^{-ky}\cos kx - e^{-Ky}\cos Kx}{k-K}\right) \tag{A 11}$$

$$= \sum k^{2s}\frac{r^{2s+1}}{(2s+1)!}\cos(2s+1)\theta + \sum\frac{(-r)^s}{s!}\frac{k^s-K^s}{k-K}\cos s\theta$$

$$= \sum k^{2s}\frac{r^{2s+1}}{(2s+1)!}\cos(2s+1)\theta + \sum\frac{(-r)^s}{s!}(k^{s-1}+k^{s-2}K+\ldots+K^{s-1})\cos s\theta$$

$$= \sum_{s=1}^{\infty}\frac{k^{2s-2}(k+K)}{K^{2s}}\left(\frac{(Kr)^{2s}}{(2s)!}\cos 2s\theta - \frac{(Kr)^{2s+1}}{(2s+1)!}\cos(2s+1)\theta+\ldots\right)$$

$$= -\frac{1}{2}\sum_{s=1}^{\infty}\frac{k^{2s-2}(k+K)}{(2s-1)!}\alpha_s(x,y).$$

On substituting this expression for (A 11) in the integral (A 9) and using (A 6) it is seen that (A 9) is equal to $\Sigma_1^\infty \alpha_s(x,y)\,\Phi_s(\xi,\eta;\,\infty)$, and similarly it can be shown that (A 10) is equal to $\Sigma_1^\infty \beta_s(x,y)\,\Psi_s(\xi,\eta;\,\infty)$. This proves the validity of (A 8) when $y < \eta$, but it is readily seen that both sides of (A 8) are regular harmonic when $|x+iy| < |\xi+i\eta|$. Thus, by analytic continuation the result (A 8) is valid for all (x, y) and (ξ, η) such that $|x+iy| < |\xi+i\eta|$. This concludes the proof of theorem A 2.

Proof of theorem A 1. In this section take C to be a semicircular arc with centre (0, 0), lying inside the strip $-\infty < x < \infty$, $0 < y < h$. Denote by A_- the semicircular domain bounded by C and the x axis, and denote by A the part of the strip exterior to C. Take the point (X, Y) in A_-, and the point (ξ, η) in A. In the domain A_- apply Green's theorem to the functions $G_0(x,y;\xi,\eta;h)$ and $G_0(x,y;X,Y;\infty)$. In A_- the former function is regular, the latter has a source singularity. Thus

$$2\pi G_0(X,Y;\xi,\eta;h)$$
$$=\int\left\{G_0(x,y;\xi,\eta;h)\frac{\partial}{\partial n}G_0(x,y;X,Y;\infty)-G_0(x,y;X,Y;\infty)\frac{\partial}{\partial n}G_0(x,y;\xi,\eta;h)\right\}ds(x,y),$$

where the integration is along the boundary of A_-. The functions $G_0(\ )$ both satisfy the free-surface condition on the upper horizontal boundary of A_-. Thus the integrand vanishes there, and we see that

$$2\pi G_0(X,Y;\xi,\eta;h)$$
$$=\int_C\left\{G_0(x,y;\xi,\eta;h)\frac{\partial}{\partial n}G_0(x,y;X,Y;\infty)-G_0(x,y;X,Y;\infty)\frac{\partial}{\partial n}G_0(x,y;\xi,\eta;h)\right\}ds \tag{A 12}$$

$= [G_0(\ldots;\xi,\eta;h), G_0(\ldots;X,Y;\infty)]$ in the notation of appendix B below

$$=\sum_0^\infty \alpha_m(X,Y)\,[G_0(\ldots;\xi,\eta;h),\Phi_m(\ldots;\infty)] \tag{A 13}$$

$$+\sum_0^\infty \beta_m(X,Y)\,[G_0(\ldots;\xi,\eta;h),\Psi_m(\ldots;\infty)] \quad \text{by theorem A 2.} \tag{A 14}$$

To evaluate the bilinear products in (A 13) and (A 14), observe that

$$\Phi_m(\ldots;\infty)-\Phi_m(\ldots;h)$$

is a regular wave function in A_-, whence by Green's theorem

$$[G_0(\ldots;\xi,\eta;h),\Phi_m(\ldots;\infty)] = [G_0(\ldots;\xi,\eta;h),\Phi_m(\ldots;h)]. \tag{A 15}$$

The last expression can be evaluated by applying Green's theorem in the domain A to the functions $\Phi_m(x,y;h)$ and $G_0(x,y;\xi,\eta;h)$. In A the former function is regular; the latter has a source singularity. There is no contribution to the line integral from the upper and lower horizontal boundaries of A, where the integrand vanishes, or from ∞, where Φ_m and G_0 represent outgoing waves.

Thus, by Green's theorem,

$$[G_0(\ldots;\xi,\eta;h),\Phi_m(\ldots;h)] = 2\pi\Phi_m(\xi,\eta;h),$$

and similarly

$$[G_0(\ldots;\xi,\eta;h),\Psi_m(\ldots;h)] = 2\pi\Psi_m(\xi,\eta;h).$$

Thus, from (A 13) and (A 14),

$$2\pi G_0(X,Y;\xi,\eta;h) = 2\pi\sum_0^\infty \alpha_m(X,Y)\,\Phi_m(\xi,\eta;h)+2\pi\sum_0^\infty \beta_m(X,Y)\,\Psi_m(\xi,\eta;h).$$

This concludes the proof of theorem A 1. From the proof the result is seen to be valid when $|X+iY| < |\xi+i\eta|$. It is not necessary for the semicircle C to lie in the strip where $0 < y < h$, but we shall not need this property.

Appendix B. Bilinear products

Suppose that the functions $f(x,y)$ and $F(x,y)$ satisfy

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f(x,y) = 0, \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)F(x,y) = 0 \tag{B 1}$$

in the strip $-\infty < x < \infty$, $0 < y < h$, that they are bounded except possibly at $(0,0)$, and that they satisfy the boundary conditions

$$Kf+\frac{\partial f}{\partial y} = 0, \quad KF+\frac{\partial F}{\partial y} = 0 \quad \text{on} \quad y = 0, \tag{B 2}$$

where $K > 0$ is real. Let the semicircle C, the inner domain A_- and the outer domain A be defined as in appendix A. Then, by definition, the bilinear product is

$$[f,F] = \int_C \left(f\frac{\partial F}{\partial n} - F\frac{\partial f}{\partial n}\right) ds, \tag{B 3}$$

where $\partial/\partial n$ denotes differentiation along the normal from C into A. It is readily seen that the value of $[f, F]$ is unaltered when the path of integration is replaced by any other path lying in the strip and extending from the positive x axis to the negative x axis. This follows from Green's theorem since f and F are regular harmonic between the paths, and since $f\,\partial F/\partial y - F\,\partial f/\partial y = 0$ on $y = 0$, from (B 2).

The calculation of $[u, u^*]$ in §3 above involves the values of $[f, F^*]$, where the asterisk denotes the complex conjugate, and where f and F may be any of the functions $\alpha_m(x,y)$, $\beta_m(x,y)$, $\Phi_m(x,y)$, $\Psi_m(x,y)$. We begin, however, by considering the values of $[\alpha_l, \Phi_m]$ and $[\beta_l, \Psi_m]$.

THEOREM B 1. *We have*

$$[\alpha_l, \Phi_l] = [\beta_l, \Psi_l] = 2\pi,$$
$$[a_l, \Phi_m] = [\beta_l, \Psi_m] = 0, \quad \textit{when} \quad l \neq m,$$
$$[\alpha_l, \Psi_m] = [\beta_l, \Phi_m] = 0.$$

Proof of theorem B 1. Apply Green's theorem in A_- to the functions $\alpha_l(x,y)$ and $G_0(x,y; X,Y; h)$, where (X, Y) lies in A_-. In A_- the former function is regular, the latter has a source singularity. We thus find that

$$\begin{aligned}2\pi\alpha_l(X, Y) &= [\alpha_l(\ldots), G_0(\ldots; X, Y; h)] \\ &= [\alpha_l(\ldots), \Sigma\alpha_m(X, Y)\,\Phi_m(\ldots; h) + \Sigma\beta_m(X,Y)\,\Psi_m(\ldots;)] \quad \text{from theorem A 1} \\ &= \Sigma\alpha_m(X, Y)\,[\alpha_l, \Phi_m] + \Sigma\beta_m(X, Y)\,[\alpha_l, \Psi_m].\end{aligned} \tag{B 4}$$

It is evident, however, that near $(X, Y) = (0,0)$ any regular wave function has a unique expansion in terms of the sets of regular wave potentials $\alpha_m(X, Y)$ and $\beta_m(X, Y)$. Thus, in particular, from (B 4)

$$[\alpha_l, \Phi_l] = 2\pi; \quad [\alpha_l, \Phi_m] = 0, \quad l \neq m; \tag{B 5}$$

and

$$[\alpha_l, \Psi_m] = 0. \tag{B 6}$$

Similarly

$$[\beta_l, \Psi_l] = 2\pi; \quad [\beta_l, \Psi_m] = 0, \quad l \neq m; \tag{B 7}$$

and

$$[\beta_l, \Phi_m] = 0. \tag{B 8}$$

This concludes the proof of theorem B 1.

The values of $[\alpha_l, \Phi_m^*]$ and $[\Phi_m, \alpha_m^*]$ are readily deduced from (B 5) by noting that α_l is real, and that $[F, f^*] = -[f^*, F] = -[f, F^*]^*$. Similarly all the bilinear products related to (B 6), (B 7) and (B 8) can be evaluated. To evaluate $[\alpha_l, \alpha_m^*]$, contract the contour of integration to the origin; it follows that $[\alpha_l, \alpha_m^*] = 0$. Similarly

$$[\alpha_l, \beta_m^*] = [\beta_l, \beta_m^*] = 0.$$

It only remains to evaluate $[\Phi_l, \Phi_{ml}^*]$ and similar terms. Consider

$$[\Phi_l(\ldots; h), \Phi_m^*(\ldots; h)] = \int_C \left(\Phi_l \frac{\partial \Phi_m^*}{\partial n} - \Phi_m^* \frac{\partial \Phi_l}{\partial n}\right) ds.$$

To evaluate this, consider

$$\int \left(\Phi_l \frac{\partial \Phi_m^*}{\partial n} - \Phi_m^* \frac{\partial \Phi_l}{\partial n}\right) ds$$

along the boundary of the exterior domain A closed by vertical lines at $x = \pm\infty$. This integral vanishes since Φ_l and Φ_m^* are regular in A. Also the integrand vanishes on $y = 0$ and $y = h$. It follows that

$$\int_C = -\int_{x=+\infty} - \int_{x=-\infty},$$

and the latter integrals can be found by using the appropriate asymptotic forms from appendix A and carrying out an elementary integration. The results are collected in the following theorem.

THEOREM B 2. *Consider the bilinear products $[f, F^*]$, where f and F may be any of the functions $\alpha_m, \beta_m, \Phi_m, \Psi_m$. Then all these bilinear products vanish except the following:*

$$[\Phi_0, \Phi_0^*] = -\frac{4\pi^2 i \cosh^2 k_0 h}{2k_0 h + \sinh 2k_0 h},$$

$$[\Phi_0, \Phi_l^*] = [\Phi_l, \Phi_0^*] = -\frac{4\pi^2 i k_0^{2l}}{(2l-1)!\,(2k_0 h + \sinh 2k_0 h)}, \quad l \geqslant 1;$$

$$[\Phi_m, \Phi_l^*] = -\frac{4\pi^2 i k_0^{2l+2m}}{(2l-1)!\,(2m-1)!\cosh^2 k_0 h\,(2k_0 h + \sinh 2k_0 h)}, \quad l \geqslant 1, \quad m \geqslant 1.$$

$$[\Psi_0, \Psi_0^*] = -\frac{4\pi^2 i k_0^2 \cosh^2 k_0 h}{2k_0 h + \sinh 2k_0 h};$$

$$[\Psi_0, \Psi_l^*] = [\Psi_l, \Psi_l^*] = -\frac{4\pi^2 i k_0^{2l+2}}{(2l)!\,(2k_0 h + \sinh 2k_0 h)}, \quad l \geqslant 1;$$

$$[\Psi_m, \Psi_l^*] = -\frac{4\pi^2 i k_0^{2l+2m+2}}{(2l)!\,(2m)!\cosh^2 k_0 h (2k_0 h + \sinh 2k_0 h)}, \quad l \geqslant 1, \quad m \geqslant 1.$$

$$[\alpha_l, \Phi_l^*] = -[\Phi_l, \alpha_l^*] = 2\pi; \quad [\beta_l, \Psi_l^*] = -[\Psi_l, \beta_l^*] = 2\pi.$$

These results are applied in theorem B 3.

THEOREM B 3. *Suppose that the wave potential $\phi(x, y)$, satisfying* (2.1) *and* (2.2), *has an expansion*

$$\phi(x,y) = \sum_0^\infty A_m \alpha_m(x,y) + \sum_0^\infty B_m \beta_m(x,y) + \sum_0^M C_m \Phi_m(x,y;h) + \sum_0^N D_m \Psi_m(x,y;h)$$

inside a domain A_- bounded by a semicircular arc C with its centre at $(0,0)$. Then

$$\frac{i}{4\pi}\int_C \left(\phi\frac{\partial\phi^*}{\partial n}-\phi^*\frac{\partial\phi}{\partial n}\right)ds$$

$$\equiv \frac{i}{4\pi}[\phi,\phi^*]$$

$$= \sum_0^M \operatorname{Im}(C_l A_l^*)+\sum_0^N \operatorname{Im}(D_l B_l^*)$$

$$+\frac{\pi}{2k_0h+\sinh 2k_0h}\left|C_0\cosh k_0h+\frac{1}{\cosh k_0h}\sum_1^M\frac{C_l k_0^{2l}}{(2l-1)!}\right|^2$$

$$+\frac{\pi k_0^2}{2k_0h+\sinh 2k_0h}\left|D_0\cosh k_0h+\frac{1}{\cosh k_0h}\sum_1^N\frac{D_l k_0^{2l}}{(2l)!}\right|^2,$$

where k_0 is the positive root of $k_0\tanh k_0h = K$.

Proof. Using obvious symmetry properties we have

$$[\phi,\phi^*] = [\Sigma A_j\alpha_j+\Sigma C_j\Phi_j, \Sigma A_l^*\alpha_l^*+\Sigma C_l^*\Phi_l^*]+[\Sigma B_j\beta_j+\Sigma D_j\Psi_j, \Sigma B_l^*\beta_l^*+\Sigma D_l^*\Psi_l^*] \quad \text{(B 9)}$$

$$= \Sigma\Sigma A_jA_l^*[\alpha_j,\alpha_l^*]+\Sigma\Sigma A_jC_l^*[\alpha_j,\Phi_l^*]+\Sigma\Sigma C_jA_l^*[\Phi_j,\alpha_l^*]+\Sigma\Sigma C_jC_l^*[\Phi_j,\Phi_l^*]$$
$$+\text{similar terms from (A 10).} \quad \text{(B 10)}$$

On using the values of the bilinear products given in theorem B 2 the result follows immediately.

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